Math 211 - Bonus Exercise 5 (please discuss on Forum)

1) Prove that if $\{G_1, \ldots, G_n\}$ and $\{G'_1, \ldots, G'_n\}$ denote the same collection of abelian groups, but perhaps in a different order, then there exists an isomorphism

$$G_1 \times \cdots \times G_n \cong G'_1 \times \cdots \times G'_n$$

Use this to construct an action of the symmetric group S_n on $G \times \cdots \times G$ (the n-fold direct product of an abelian group G) by automorphisms.

2) We have seen that any short exact sequence of abelian groups $0 \to K \to G \to \mathbb{Z}^r \to 0$ splits. But is it also true that any short exact sequence of abelian groups $0 \to \mathbb{Z}^r \to G \to L \to 0$ also splits?

3) Any homomorphism

$$f : \mathbb{Z}^r \to \mathbb{Z}^r$$

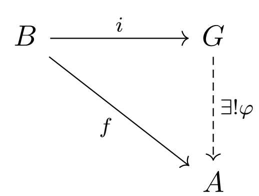
can be completely determined by a $r \times r$ matrix A with integer coefficients, by

$$f\left(\begin{pmatrix} k_1 \\ \vdots \\ k_r \end{pmatrix}\right) = A \begin{pmatrix} k_1 \\ \vdots \\ k_r \end{pmatrix}$$

(we write tuples $(k_1, \ldots, k_r)$ in column form). Which property purely in terms of A tells you if f is injective? What about the index of $\mathrm{Im}\, f$ in $\mathbb{Z}^r$, can that be expressed purely in terms of A?

4) Prove that if G is a finitely generated abelian group, then it is not divisible: this means that there exists some $g \in G$ and some $n > 0$ such that $\frac{g}{n}$ does not exist in G (i.e. $g \neq nh, \forall h \in G$).

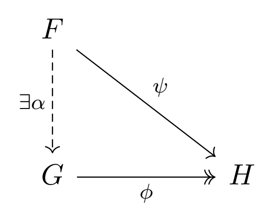
5) Show that an abelian group G is free if and only if there exists a subset $B \subset G$ satisfying the following universal property: for all abelian groups A and all set function $f : B \to A$, there exists a unique homomorphism of groups $\varphi : G \to A$ such that $\varphi \circ i = f$:



where $i : B \to G$ is the set inclusion. Note that the subset B is a **basis** of the free abelian group G.

6)Show that if $\mathbb{Z}^n \cong \mathbb{Z}^m$, then $n = m$.
Hint: You know from linear algebra that this statement is true if instead of $\mathbb{Z}$ we had a field k and a k - linear isomorphism. Can you reduce to this case by quotienting by appropriate subgroups?

7) Prove that a finitely generated abelian group F is free if and only if for all pairs (ϕ, ψ), where $\phi : G \to H$ is a surjective homomorphism between two abelian groups G and H, and $\psi : F \to H$ is a homomorphism, there exists a homomorphism $\alpha : F \to G$ such that $\psi = \phi \circ \alpha$:



Note: We call an abelian group which satisfies the above property projective. This exercise proves that being projective is the same as being free in the case of finitely generated abelian groups.